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Randomly Flashing Diffusion: Asymptotic Properties

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The theory of abstract Markov operators and semigroups is applied for studying asymptotics of a randomly flashing diffusion process. The probability distribution of the process is determined by a set of two partial differential equations and sufficient conditions for the existence of a stationary solution of the equations are formulated, and convergence of solutions to the stationary solution is proved.

KEY WORDS: Diffusion; Markov semigroups; asymptotic stability.

1. INTRODUCTION

A great deal of effort has been devoted to the study of the influence of noise and random perturbations on systems described by differential equations of the Langevin type. Usually, processes driven by a single noise have been investigated, while *composite noises*, the product of several elementary noises, have rather seldom been considered. Multistate diffusion processes and multistate random walks[(1)] are examples of process with composite noises. A two-state diffusion process is considered in refs. 2 and 3. For this process, a Brownian motion jumps between a diffusion process of strength D_1 and a diffusion process of strength D_2. The sojourn time in each state is random and transitions between two states occur by means of a dichotomic Markovian process[(2)] or Poission process.[(3)] A special case of the two-state process, i.e., when $D_2 = 0$, is analyzed in refs. 4–7. Processes of this kind are termed randomly flashing or interrupted diffusion because a system jump between diffusional ($D_1 \neq 0$) and deterministic ($D_2 = 0$) states and jumps are driven by a random process.

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In this paper we study the asymptotic behavior of probability distributions of the randomly flashing diffusion x_t described by the following stochastic equation (with the Stratonovich interpretation[8]):

$$\dot{x}_t = f(x_t) + g(x_t)\, \Gamma(t), \qquad \dot{x}_t = \frac{d}{dt} x_t \tag{1.1}$$

where $f(x)$ and $g(x)$ are deterministic functions, and $\Gamma(t)$ is a composite stochastic process of the form

$$\Gamma(t) = \xi(t)\, \eta(t) \tag{1.2}$$

The process $\eta(t)$ is a Gaussian white noise with the first two moments

$$\langle \eta(t) \rangle = 0, \qquad \langle \eta(t)\, \eta(s) \rangle = 2\delta(t-s) \tag{1.3}$$

and $\xi(t)$ is a two-state $\{1, 0\}$ Markov process with transition probabilities $1 \to 0$ and $0 \to 1$ in small time dt equal to $a\,dt$ and $b\,dt$, $a, b > 0$, respectively. Initial probabilities for $\xi(t)$ are chosen arbitrarily. We assume that $\eta(t)$ and $\xi(t)$ are independent stochastic processes and the initial value x_0 is independent of $\Gamma(t)$. Notice that the noise $\Gamma(t)$ is a Dirac delta-correlated (or uncorrelated) process as in (1.3), but generally with the time-dependent intensity $D(t) = 2\langle \xi^2(t) \rangle$.

The noise $\Gamma(t)$ in (1.1) might be interpreted as a Langevin force switched on and off at random time instants with Poisson statistics. If t is interpreted as a spatial variable, then $\Gamma(t)$ can model a stochastic two-layer medium: one layer is a medium with a diffusion coefficient $D_1 = 1$,[4,9] and the other is the vacuum (surroundings characterized by a diffusion coefficient $D_2 = 0$). Equation (1.1) is an example of two-state models in which transitions from one state [deterministic: $\dot{x}_t = f(x_t)$] to the other state [diffusional: $\dot{x}_t = f(x_1) + g(x_t)\, \eta(t)$] and vice versa occur at random moments. Models of this type are considered in ref. 10. As possible applications of equations like (1.1) one could mention the problem of multiple scattering of particles through plates of matter separated by vacuum gaps,[4] transport phenomena in sponge-type strutures with empty places (vacua) and matter randomly distributed in space, and wave propagation in randomly stratified media.[11] Moreover, models of randomly flashing diffusion might find an application in domains where information is transcoded along tottery transmission lines, e.g., in radiophysics, or in faulty neuron networks. Additionally, flashing diffusion models might also find applications for control of chaos[12] or phase control.[13]

A closed form of an evolution equation for a one-dimensional probability density $p(x, t)$ of the process (1.1) is derived in ref. 5. It is rather a complicated partial integrodifferential equation. For a linear noise-additive case, when $f(x) = -ax$, $a > 0$, and $g(x) = 1$, this equation is solved in ref. 6 and analyzed in detail in ref. 7. The first-passage-time problem for the linear model is studied in ref. 14, where it is shown that boundary and natural conditions for integration of differential equations determining the mean first passage time depend strongly on the domain which the process x_t is to exit. A simplified case, when $f(x) = 0$ and $g(x) = 1$, is investigated in ref. 4. In this paper, we consider in general *nonlinear processes* defined by (1.1) with arbitrary functions $f(x)$ and $g(x)$.

A pair $(x_t, \xi(t))$ constitutes a Markov process on $R \times \{1, 0\}$. Equations for probability distributions $p_1(x, t)$ and $p_0(x, t)$ of this process [having values $(x, 1)$ and $(x, 0)$, respectively] are of the form[15, 16]

$$\begin{cases} \dfrac{\partial p_1}{\partial t} = -ap_1 + bp_0 - \dfrac{\partial}{\partial x}(fp_1) + \dfrac{\partial}{\partial x}\left(g\dfrac{\partial}{\partial x}(gp_1)\right) \\ \dfrac{\partial p_0}{\partial t} = ap_1 - bp_0 - \dfrac{\partial}{\partial x}(fp_0) \end{cases} \tag{1.4}$$

The probability distribution $p(x, t)$ of the process x_t alone is given by $p(x, t) = p_1(x, t) + p_0(x, t)$.

The purpose of this paper is to give sufficient conditions for the existence of stationary states of the process (1.1), a stationary solution of the system (1.4), and convergence of solutions to the stationary solution. Such a property of the system (1.4) is called asymptotic stability and it fully describes the behavior of the system as $t \to \infty$. It is formulated as a theorem in Section 2. The main idea of the paper is to apply the theory of abstract Markov operators and semigroups to study the system (1.4). This approach to the study of the asymptotic statibility has been recently applied to such diverse areas as diffusion processes, astrophysics, queuing theory, and population dynamics.[17-20] First, using the Phillips perturbation theorem,[1] we give a formula for solutions of (1.4) in Section 3. Next, in Section 4, we formulate a criterion for asymptotic stability of Markov semigroups. Then, we show that the system (1.4) is asymptotically stable if and only if it has a stationary solution. Finally, we prove the existence of the stationary solution (Section 5). Final remarks are given in Section 6.

2. MAIN RESULT

Throughout the paper we assume that $f \in C^2(R)$, $g \in C^2(R)$, and $g(x) \geq c > 0$, where $C^2(R)$ is the space of two-times-differentiable bounded functions whose derivatives of order ≤ 2 are continuous and bounded. Since the solutions of the system (1.4) are probability distributions of some

Markov process, we assume that $p_1(x, t) \geqslant 0$, $p_0(x, t) \geqslant 0$ and the normalization condition is fulfilled,

$$\int_{-\infty}^{\infty} [p_1(x, t) + p_0(x, t)]\, dx = 1$$

The main result of this paper is the following:

Theorem 1. Let

$$F(x) = \int_0^x \frac{f(z)\, dz}{g^2(z)}$$

and assume that:

(i) There exists $x_0 > 0$ such that $xf(x) < 0$ for $|x| \geqslant x_0$.

(ii) $\lim_{|x| \to \infty} |f(x)|\, e^{-F(x)} = \infty$.

Then there exists a stationary solution $p^*(x) = (p_1^*(x), p_0^*(x))$ of the system (1.4). For every solution $(p_1(x, t), p_0(x, t))$ of (1.4) we have

$$\lim_{t \to \infty} \int_{-\infty}^{\infty} \{|p_1(x, t) - p_1^*(x)| + |p_0(x, t) - p_0^*(x)|\}\, dx = 0 \tag{2.1}$$

There proof of Theorem 1 is given in Section 5. Here we only make the reduction to the special case $g(x) = 1$ and $a = 1$. Setting

$$h(x) = \int_0^x \frac{\sqrt{a}\, dr}{g(r)} \qquad \text{and} \qquad \bar{f}(x) = \frac{f(h^{-1}(x))}{\sqrt{a}\, g(h^{-1}(x))}$$

and substituting

$$p_1(x, t) = h'(x)\, \bar{p}_1(h(x), at), \qquad p_0(x, t) = h'(x)\, \bar{p}_0(h(x), at)$$

(the prime denotes a derivative with respect to x), we obtain

$$\begin{cases} \dfrac{\partial \bar{p}_1}{\partial t} = -\bar{p}_1 + \dfrac{b}{a} \bar{p}_0 - \dfrac{\partial}{\partial x} (\bar{f} \bar{p}_1) + \dfrac{\partial^2 \bar{p}_1}{\partial x^2} \\ \dfrac{\partial \bar{p}_0}{\partial t} = \bar{p}_1 - \dfrac{b}{a} \bar{p}_0 - \dfrac{\partial}{\partial x} (\bar{f} \bar{p}_0) \end{cases} \tag{2.2}$$

An easy computation shows that (ii) holds if and only if

$$\lim_{|x| \to \infty} |\bar{f}(x)| \exp\left[- \int_0^x \bar{f}(r)\, dr \right] = \infty \tag{2.3}$$

Since the operator $Hv(x) = h'(x)\, v(h(x))$ preserves the norm in $L^1(R)$, the system (1.4) is asymptotically stable if and only if the system (2.2) is asymptotically stable. From now on we consider the system (1.4) with $g(x) = 1$ and $a = 1$.

3. SEMIGROUP REPRESENTATION OF SOLUTIONS

In this section we give a formula for the solutions of the system

$$\begin{cases} \dfrac{\partial p_1}{\partial t} = -p_1 + bp_0 - \dfrac{\partial}{\partial x}(fp_1) + \dfrac{\partial^2 p_1}{\partial x^2} \\ \dfrac{\partial p_0}{\partial t} = p_1 - bp_0 - \dfrac{\partial}{\partial x}(fp_0) \end{cases} \tag{3.1}$$

Denote by A and B the linear operators

$$Av(x) = -\frac{d}{dx}(fv) + \frac{d^2 v}{dx^2}, \qquad Bv(x) = -\frac{d}{dx}(fv)$$

Operators A and B generate continuous semigroups of Markov operators $\{T^+(t)\}_{t\geqslant 0}$ and $\{T^-(t)\}_{t\geqslant 0}$ on the space $L^1(R)$. Formally, for any $v \in L^1(R)$ the functions $u_+(t) = T^+(t)v$ and $u_-(t) = T^-(t)v$ are the solutions of the evoluation equations $u'_+(t) = Au_+$ and $u'_-(t) = Bu_-$ with the initial conditions $u_+(0) = u_-(0) = v$ (the prime denotes a derivative with respect to t). Now, let λ be a constant such that $\lambda > \max\{1, b\}$. We define the operators $\mathscr{A}$ and $\mathscr{T}$ by

$$\mathscr{A}(p_1, p_0) = (Ap_1, Bp_0)$$

$$\mathscr{T}(p_1, p_0) = \lambda^{-1}((\lambda - 1)\, p_1 + bp_0,\ p_1 + (\lambda - b)\, p_0)$$

Then instead of the system (3.1) we can consider one evolution equation:

$$p'(t) = \lambda \mathscr{T} p - \lambda p + \mathscr{A} p \tag{3.2}$$

where $p = (p_1, p_0)$. The operator $\mathscr{A}$ generates a semigroup $\{T(t)\}_{t\geqslant 0}$ of operators on the space $L^1(R) \times L^1(R)$ given by the formula

$$T(t)(p_1, p_0) = (T^+(t)\, p_1,\ T^-(t)\, p_0)$$

Let $X = R \times \{1, 0\}$ and $L^1(X)$ be the Banach space of integrable functions on X with the norm

$$\|v\| = \int |v(x, 1)|\, dx + \int |v(x, 0)|\, dx$$

We can identify the spaces $L^1(R) \times L^1(R)$ and $L^1(X)$ by $v_i(x) = v(x, i)$ for $i = 1, 0$ and $v \in L^1(X)$. Then $\{T(t)\}_{t \geqslant 0}$ is a semigroup of Markov operators on the space $L^1(X)$ and $\mathscr{T}$ is a Markov operator on $L^1(X)$.

From the Phillips perturbation theorem[21] Eq. (3.2) with the initial condition $p(0) = v$ generates a continuous semigroup $\{S(t)\}_{t \geqslant 0}$ of Markov operators on $L^1(X)$ given by

$$S(t)v = p(t) = e^{-\lambda t} \sum_{n=0}^{\infty} \lambda^n T_n(t) v \tag{3.3}$$

where $T_0(t) = T(t)$ and

$$T_{n+1}(t)v = \int_0^t T_0(t-s)\, \mathscr{T} T_n(s) v \, ds, \qquad n \geqslant 0 \tag{3.4}$$

Thus, instead of studying solutions of Eq. (3.1), we study the behavior of the semigroup $\{S(t)\}_{t \geqslant 0}$. Let $D(X)$ denote the set of all probability densities on $L^1(X)$, i.e.,

$$D(X) = \left\{ v \in L^1(X) \colon v \geqslant 0, \int v(x)\, dx = 1 \right\}$$

The density (or measure) v_* is called invariant under the semigroup $\{S(t)\}_{t \geqslant 0}$ if $S(t)v_* = v_*$ for every $t \geqslant 0$. The semigroup $\{S(t)\}_{t \geqslant 0}$ is called asymptotically stable if it has an invariant density v_* and for every $v \in D(X)$

$$\lim_{t \to \infty} \|S(t)v - v_*\| = 0$$

Now, the condition (2.1) is equivalent to the asymptotic stability of the semigroup $\{S(t)\}_{t \geqslant 0}$.

Since in formulas (3.3) and (3.4) the essential role is played by semigroups $\{T^+(t)\}_{t \geqslant 0}$ and $\{T^-(t)\}_{t \geqslant 0}$, we give some auxiliary results on these semigroups.

The semigroup $\{T^+(t)\}_{t \geqslant 0}$ is an integral semigroup, i.e., for every $t > 0$ there exists a Borel measurable function $k_t \colon R \times R \to R$ such that

$$T^+(t)\, v(x) = \int_{-\infty}^{\infty} k_t(x, y)\, v(y)\, dy \tag{3.5}$$

The function $k_t(x, y)$ has some additional properties: it is strictly positive and continuous with respect to (t, x, y) and there exists a constant $m > 0$ such that

$$k_t(x, y) \leqslant m(1 + t^{-1/2}) \qquad \text{for} \quad t > 0, \quad x \in R, \quad y \in R \tag{3.6}$$

The semigroup $\{T^-(t)\}_{t \geqslant 0}$ can be explicitly given. Namely, the function $u(x, t) = T^-(t)\, v(x)$ is a solution of the equation

$$\frac{\partial u}{\partial t} = -\frac{\partial}{\partial x}(f(x)u) \tag{3.7}$$

with the initial condition

$$u(0, x) = v(x)$$

Equation (3.7) can be solved by the method of characteristics. For each $\bar{x} \in R$ denote by $\pi_t \bar{x}$ the solution $x(t)$ of the equation

$$x'(t) = f(x(t))$$

with the initial condition $x(0) = \bar{x}$. Then

$$T^-(t)\, v(x) = u(x, t) = v(\pi_{-t}\, x) \frac{\partial}{\partial x}(\pi_{-t} x) \tag{3.8}$$

The formula (3.8) can be written down in the following way:

$$T^-(t)\, v(x) = \begin{cases} (v(\pi_{-t} x)\, f(\pi_{-t} x))/f(x) & \text{if} \quad f(x) \neq 0 \\ v(x) & \text{if} \quad f(x) = 0 \end{cases} \tag{3.9}$$

4. CONDITIONS FOR ASYMPTOTIC STABILITY

First, we formulate a criterion for the asymptotic stability of Markov semigroups. Let $(Y, \mathscr{A}, \mu)$ be a σ-finite measure space. Denote by $D = D(Y)$ the subset of $L^1(Y)$ which consist of all densities. A semigroup $\{P(t)\}_{t \geqslant 0}$ of linear operators on $L^1(Y)$ is called a Markov semigroup if $P(t)(D) \subset D$ for every $t \geqslant 0$.

A Markov semigroup $\{P(t)\}_{t \geqslant 0}$ is called partially integral if for $t > 0$ the operator $P(t)$ can be written in the form

$$P(t)\, v(x) = \int_Y q_t(x, y)\, v(y)\, \mu(dy) + R(t)\, v(x)$$

where $R(t)$ is a nonnegative operator on $L^1(Y)$ and $q_t(x, y)$ is a measurable nonnegative function such that

$$\int_Y q_t(x, y)\, \mu(dy) > 0 \qquad \text{for} \quad x \in Y$$

As in Section 3, a density v_* is called invariant if $P(t) v_* = v_*$ for every $t \geqslant 0$. The semigroup $\{P(t)\}_{t \geqslant 0}$ is asymptotically stable if it has an invariant density v_* and $\lim_{t \to \infty} \|P(t) v - v_*\| = 0$ for every v. The support of a $v \in L^1(Y)$ is defined up to a set of measure zero by the formula

$$\text{supp } v = \{x \in Y \colon v(x) \neq 0\}$$

Proposition 1. Assume that the partially integral semigroup $\{P(t)\}_{t \geqslant 0}$ has an invariant density. If there exists $t > 0$ such that for every $v \in D(Y)$ we have

$$\text{supp } P(t) v = Y \tag{4.1}$$

then the semigroup $\{P(t)\}_{t \geqslant 0}$ is asymptotically stable.

Proposition 1 is a simple consequence of the Theorem 1 in ref. 22. We only check that the semigroup $\{S(t)\}_{t \geqslant 0}$ generated by the system (3.1) satisfies the assumptions of Proposition 1. Since the semigroup $\{T^+(t)\}_{t \geqslant 0}$ is an integral semigroup with a strictly positive kernel $k_t(x, y)$, from (3.4) it follows immediately that the semigroup $\{S(t)\}_{t \geqslant 0}$ is partially integral. We check that the semigroup $\{S(t)\}_{t \geqslant 0}$ satisfies the condition (4.1). Indeed, let $u \in L^1(R)$ be such that $u \geqslant 0$ and $\int u(x)\, dx > 0$. Then, since $\{T^+(t)\}_{t \geqslant 0}$ is an integral semigroup with a strictly positive kernel, we have supp $T^+(t) u = R$. Moreover, from (3.8) it follows that if supp $u = R$, then supp $T^-(t) u = R$. Now, let $v \in D(X)$. Then from the above observations and from (3.4) it follows that supp $T_2(t) v = X$ for $t > 0$. This implies that supp $S(t) v = X$ for $t > 0$ and $v \in D$.

Now, let v_* be an invariant density with respect to semigroup $\{S(t)\}_{t \geqslant 0}$. Then the function $p(x) = (p_1(x), p_0(x))$, where $p_1(x) = v_*(x, 1)$ and $p_0(x) = v_*(x, 0)$, is a stationary solution of the system (3.1). By use of operators A and B we write (3.1) in the following way:

$$\begin{cases} b p_0 = (I - A)\, p_1 \\ \ p_1 = (bI - B)\, p_0 \end{cases} \tag{4.2}$$

Let $R(\lambda, A)$ and $R(\lambda, B)$ be the resolvents of the operators A and B, respectively. Put $R(A) = R(1, A)$ and $R(B) = bR(b, B)$. Then (4.2) is equivalent to the system

$$\begin{cases} p_1 = bR(A)\, p_0 \\ bp_0 = R(B)\, p_1 \end{cases} \tag{4.3}$$

It is easy to check that $R(A)$ and $R(B)$ are Markov operators on the space $L^1(R)$.

Let $P = R(A)\, R(B)$. Then P is also a Markov operator on $L^1(R)$. If the operator P has an invariant density $v \in L^1(R)$, then the functions

$$p_1 = \frac{b}{b+1}\, v, \qquad p_0 = \frac{1}{b+1}\, R(B)\, v$$

are solutions of (4.3).

In order to prove the Theorem 1, it remains to check that conditions (i) and (ii) imply the existence of an invariant density with respect to P. This will be done in the next section.

5. PROOF OF THEOREM 1

In the proof of Theorem 1 we use some auxiliary results concerning the operators $R(A)$, $R(B)$, and P.

The operator $R(A)$ is given by the formula

$$R(A)\, v(x) = \int_0^\infty e^{-t} T^+(t)\, v(x)\, dt$$

From (3.5) and (3.6) it follows that $R(A)$ is an integral operator with a kernel

$$\begin{aligned} k(x, y) &= \int_0^\infty e^{-t} k_t(x, y)\, dt \\ &\leqslant m \int_0^\infty e^{-t}(1 + t^{-1/2})\, dt = (1 + \sqrt{\pi})m \end{aligned}$$

Let $M = (1 + \sqrt{\pi})m$ and denote by D the subset of all densities in $L^1(R)$. Then

$$R(A)\, v(x) \leqslant M \tag{5.1}$$

for every $v \in D$, $x \in R$.

Since the function $k_t(x, y)$ is strictly positive and continuous with respect to (t, x, y), the kernel $k(x, y)$ has the following property. For every $y_1 < y_2$ and $x \in R$ there exists $\varepsilon > 0$ such that

$$k(x, y) \geqslant \varepsilon \qquad \text{for} \quad y \in [y_1, y_2] \tag{5.2}$$

The operator $R(B)$ can be given by an explicit formula. Since

$$R(B)\, v(x) = \int_0^{\infty} be^{-bt} T^{-}(t)\, v(x)\, dt \tag{5.3}$$

the condition (3.9) implies $R(B)\, v(x) = v(x)$ if $f(x) = 0$ and

$$R(B)\, v(x) = \int_0^{\infty} be^{-bt} \frac{v(\pi_{-t} x)\, f(\pi_{-t} x)}{f(x)}\, dt \tag{5.4}$$

if $f(x) \neq 0$. Let $\pi_{-\infty} x = \lim_{t \to \infty} \pi_{-t} x$. Then substituting $y = \pi_{-t} x$ in (5.4), we obtain

$$R(B)\, v(x) = \frac{b}{f(x)} \int_{\pi_{-\infty} x}^{x} \exp\left(\int_x^y \frac{b\, dz}{f(z)}\right) v(y)\, dy \tag{5.5}$$

Now, let v: $R \to R$ be a nonnegative measurable function. Assume that $\operatorname{supp} v \subset [x_1, x_2]$ and let $y_1 = \min\{x_1, \pi_{-1} x_1\}$, $y_2 = \max\{x_2, \pi_{-1} x_2\}$. Then $\operatorname{supp} T^{-}(t) v \subset [y_1, y_2]$ for $t \in [0, 1]$. From (5.3) it follows that

$$\begin{aligned} \int_{y_1}^{y_2} R(B)\, v(x)\, dx &\geqslant \int_0^1 be^{-bt} \int_{y_1}^{y_2} T^{-}(t)\, v(x)\, dx\, dt \\ &= \int_0^1 be^{-bt}\, dt \int_{x_1}^{x_2} v(x)\, dx \\ &= (1 - e^{-b}) \int_{x_1}^{x_2} v(x)\, dx \end{aligned}$$

In particular, if $\int_{x_1}^{x_2} v(x)\, dx = \infty$, then $\int_{y_1}^{y_2} R(B)\, v(x)\, dx = \infty$. From (5.2) it follows that if v is a nonnegative function such that $\int_{x_1}^{x_2} v(x)\, dx = \infty$, then for every $x \in R$

$$Pv(x) = R(A)\, R(B)\, v(x) \geqslant \varepsilon \int_{y_1}^{y_2} R(B)\, v(x)\, dy = \infty \tag{5.6}$$

Moreover, P is an integral operator and $Pv(x) > 0$ for every $v \in D$ and $x \in R$.

Now, we need some auxiliary definitions. A sequence of densities $\{v_n\}$ is called sweeping if

$$\lim_{n \to \infty} \int_{-c}^{c} v_n(x)\, dx = 0 \qquad \text{for every} \quad c > 0$$

A Markov operator Q: $L^1(R) \to L^1(R)$ is called sweeping if for every $v \in D$ the sequence $\{Q^n v\}$ is sweeping. If Q is an integral operator, Q has no invariant density and there exists a positive, locally integrable function v^* such that $Qv^* \leqslant v^*$, then the operator Q is sweeping. The prove of this theorem is given in ref. 23.

The following lemma shows that the operator $P = R(A)\, R(B)$ is sweeping or it has an invariant density.

Lemma 1. If the operator P has no invariant density, then P is sweeping.

Proof. The proof is based on the abstract theory of Markov processes (see refs. 22 and 24 for details). If P has no invariant density, then it is sufficient to check that there exists a positive, locally integrable function v^* such that $Pv^* \leqslant v^*$. Since $Pv(x) > 0$ for every $v \in D$ and $x \in R$, the operator P is dissipative or conservative. If P is dissipative, then for every $v \in D$, $v > 0$, we have $v^* = \sum_{n=0}^{\infty} P^n v < \infty$ and $Pv^* \leqslant v^*$. If P is conservative, then P is a Harris operator and, consequently, there exists a measurable function v^* such that $0 < v^* < \infty$ and $Pv^* = v^*$. It remains to check that v^* is locally integrable. Suppose, on the contrary, that there is a closed, bounded interval $[x_1, x_2]$ such that $\int_{x_1}^{x_2} v^*(x)\, dx = \infty$. Then from (5.6) it follows that $Pv^* = \infty$ for every $x \in R$, which is impossible.

In the next three lemmas we show that the operator P is not sweeping.

Lemma 2. Let $\gamma(x) = |\int_0^x e^{-F(t)}\, dt|$. Then for every $v \in D$ we have

$$\int_{-\infty}^{\infty} \gamma(x)\, R(A)\, v(x)\, dx \leqslant \int_{-\infty}^{\infty} \gamma(x)\, v(x)\, dx + 2M \tag{5.7}$$

Proof. Let D_0 be a dense subset of D consisting of densities with bounded supports. Since $R(A)$ is a Markov operator, it is sufficient to check (5.7) for $v \in D_0$. Let $v \in D_0$ and let $u(x, t)$ be the solution of the equation $\partial u/\partial t = Au$ with $u(x, 0) = v(x)$. Then for any $t > 0$, $c > 0$, and non-negative integers i and j we have

$$\lim_{|x| \to \infty} e^{c|x|} \frac{\partial^{i+j} u}{\partial t^i\, \partial x^j}(x, t) = 0$$

Let $\varphi(t)=\int_{-\infty}^{\infty}\gamma(x)\,u(x,t)\,dx$. Since $\gamma(x)$ grows at most exponentially, the function $\varphi(t)$ is well defined. We have

$$\varphi'(t)=\int_{-\infty}^{\infty}\gamma(x)\left[-\frac{\partial}{\partial x}(f(x)\,u(x,t))+\frac{\partial^2 u(x,t)}{\partial x^2}\right]dx$$

Integrating the last integral by parts, we obtain

$$\varphi'(t)=2u(0,t) \tag{5.8}$$

From the definition of $R(A)v$ it follows that

$$\begin{aligned}\int_{-\infty}^{\infty}\gamma(x)\,R(A)\,v(x)\,dx&=\int_0^{\infty}e^{-t}\varphi(t)\,dt\\&=\varphi(0)+\int_0^{\infty}e^{-t}\varphi'(t)\,dt\\&=\varphi(0)+2\int_0^{\infty}e^{-t}u(0,t)\,dt\\&=\int_{-\infty}^{\infty}\gamma(x)\,v(x)\,dx+2R(A)\,v(0)\\&\leqslant\int_{-\infty}^{\infty}\gamma(x)\,v(x)\,dx+2M\end{aligned}$$

Lemma 3. If a sequence $\{v_n\}$ is sweeping, then the sequence $\{R(B)\,v_n\}$ is sweeping.

Proof. Let $x_0>0$ be a constant such that $f(x)<0$ for $x>x_0$ and $f(x)>0$ for $x<-x_0$. Than for any $v\in D$ and $\alpha>x_0$ we have

$$\begin{aligned}&\int_{-\alpha}^{\alpha}R(B)\,v(x)\,dx\\&\quad=1-\int_{-\infty}^{-\alpha}\frac{b}{f(x)}\int_{-\infty}^{x}v(y)\exp\left\{\int_x^y\frac{b\,dz}{f(z)}\right\}dy\,dx\\&\qquad+\int_{\alpha}^{\infty}\frac{b}{f(x)}\int_x^{\infty}v(y)\exp\left\{\int_x^y\frac{b\,dz}{f(z)}\right\}dy\,dx\end{aligned}$$

Changing the order of integration in the last integrals, we obtain

$$\int_{-\alpha}^{\alpha}R(B)\,v(x)\,dx=\int_{-\alpha}^{\alpha}v(y)\,dy+\int_{-\infty}^{\infty}v(y)\,\beta(y)\,dy \tag{5.9}$$

where

$$\beta(y) = \exp\left\{-\int_y^{-\alpha} \frac{b\,dz}{f(z)}\right\} \qquad \text{for} \quad y < -\alpha$$

$$\beta(y) = \exp\left\{\int_\alpha^y \frac{b\,dz}{f(z)}\right\} \qquad \text{for} \quad y > \alpha$$

$$\beta(y) = 0 \qquad \text{for} \quad y \in [-\alpha, \alpha]$$

Since $f(x)$ is a bounded function, we have $\lim_{|y| \to \infty} \beta(y) = 0$. This and (5.9) imply that if $\{v_n\}$ is sweeping, then $\{R(B)\, v_n\}$ is sweeping.

Lemma 4. Let v be a density with a bounded support. Then the sequence $\{P^n v\}$ is not sweeping.

Proof. Let $w \in D$. Then $R(B) w = w + b^{-1} BR(B) w$. Since

$$\lim_{x \to \infty} e^{-F(x)} f(x) = -\infty, \qquad \lim_{x \to -\infty} e^{-F(x)} f(x) = \infty$$

there exist $M_1 > 0$ and $\alpha > 0$ such that

$$\begin{aligned} &e^{-F(x)} f(x) \operatorname{sign} x \leqslant M_1 b && \text{for} \quad x \in R \\ &e^{-F(x)} f(x) \operatorname{sign} x \leqslant (-4M - M_1) b && \text{for} \quad |x| \geqslant \alpha \end{aligned} \tag{5.10}$$

First, we check that

$$\int_{-\infty}^{\infty} \gamma(x)\, Pw(x)\, dx \leqslant \int_{-\infty}^{\infty} \gamma(x)\, w(x)\, dx - 2M + (4M + M_1) \int_{-\alpha}^{\alpha} R(B)\, w(x)\, dx \tag{5.11}$$

Since P and $R(B)$ are Markov operators, it is sufficient to check (5.11) for $w \in D_0$. If $w \in D_0$, then from (5.10) and Lemma 2. it follows that

$$\begin{aligned} &\int_{-\infty}^{\infty} \gamma(x)\, Pw(x)\, dx \\ &\quad \leqslant \int_{-\infty}^{\infty} \gamma(x)\, R(B)\, w(x)\, dx + 2M \\ &\quad = \int_{-\infty}^{\infty} \gamma(x)\, w(x)\, dx + 2M - \int_{-\infty}^{\infty} b^{-1} \gamma(x) \frac{d}{dx} (f(x)\, R(B)\, w(x))\, dx \end{aligned}$$

$$= \int_{-\infty}^{\infty} \gamma(x)\, w(x)\, dx + 2M + \int_{-\infty}^{\infty} b^{-1} e^{-F(x)} f(x) \operatorname{sign} x R(B)\, w(x)\, dx$$

$$\leqslant \int_{-\infty}^{\infty} \gamma(x)\, w(x)\, dx + 2M + M_1$$

$$- (4M + M_1) \left(\int_{-\infty}^{-\alpha} R(B)\, w(x)\, dx + \int_{\alpha}^{\infty} R(B)\, w(x)\, dx \right)$$

$$\leqslant \int_{-\infty}^{\infty} \gamma(x)\, w(x)\, dx - 2M + (4M + M_1) \int_{-\alpha}^{\alpha} R(B)\, w(x)\, dx$$

Since $\int_{-\alpha}^{\alpha} R(B)\, w(x)\, dx \leqslant 1$, the inequality (5.11) implies

$$\int_{-\infty}^{\infty} \gamma(x)\, Pw(x)\, dx \leqslant \int_{\infty}^{\infty} \gamma(x)\, w(x)\, dx + 2M + M_1 \tag{5.12}$$

If v is a density with a bounded support, then $\int_{-\infty}^{\infty} \gamma(x)\, v(x)\, dx < \infty$ and from (5.12) it follows that $\int_{-\infty}^{\infty} \gamma(x)\, P^n v(x)\, dx < \infty$ for any positive integer n. Now, we are ready to prove that $\{P^n v\}$ is not sweeping. Suppose, contrary to our claim, that $\{P^n v\}$ is sweeping. Then, according to Lemma 3, the sequence $\{R(B)\, P^n v\}$ is sweeping. This implies that

$$\int_{-\alpha}^{\alpha} R(B)\, P^n v(x)\, dx \leqslant M/(4M + M_1)$$

for sufficiently large n (say $n \geqslant n_0$). From (5.11) it follows that

$$\int_{-\infty}^{\infty} \gamma(x)\, P^{n+1} v(x)\, dx \leqslant \int_{-\infty}^{\infty} \gamma(x)\, P^n v(x)\, dx - M \qquad \text{for} \quad n \geqslant n_0$$

This implies that

$$\lim_{n \to \infty} \int_{-\infty}^{\infty} \gamma(x)\, P^n v(x)\, dx = -\infty$$

which is impossible.

From Lemma 4 it follows that the operator P is not sweeping and consequently P has an invariant density. This implies that the semigroup $\{S(t)\}_{t \geqslant 0}$ is asymptotically stable, which completes the proof of Theorem 1.

6. CONCLUDING REMARKS

We have investigated the asymptotic properties of the randomly flashing diffusion process defined by Eq. (1.1). The main result is given by

Theorem 1, where conditions for the existence of the invariant measure and limiting states of the process are established. Let us recall that for noise-driven nonlinear systems, invariant measures can play an essential role in determining noisy bifurcations and stability properties of the system. Two comments on assumptions (i) and (ii) of Theorem 1 are in order. Assumption (i) is close to the assumption that the potential $U(x)$ defined as

$$U(x) = -\int_0^x f(y)\,dy$$

is attracive for long distance (when $|x|$ is great). It is a quite natural and physical assumption for deterministic counterparts possessing stationary states. Assumption (ii) is rather technical. Nevertheless, it is close to the condition for the existence of stationary states for the corresponding (Fokker–Planck) diffusion process.

We have proved the existence of the invariant measure and stationary states $p^*(x) = (p_1^*(x), p_0^*(x))$ in the extended phase space $R \times \{1, 0\}$ of the process $(x_t, \xi(t))$. For the process x_t alone, its probability density is $p(x, t) = p_1(x, t) + p_0(x, t)$. However, the limiting distribution

$$P(x) = \lim_{t \to \infty} p(x, t) = p_1^*(x) + p_0^*(x)$$

is not, in the general case, an invariant density of an evolution operator U^t defined by the relation

$$p(x, t) = U^t P_0(x)$$

where $P_0(x)$ is the initial probability density of the process x_t. This can easily be shown for a linear model with additive noise,[6, 7] since explicit solutions $p(x, t)$ are known and limiting densities $P(x)$ are presented in ref. 7.

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